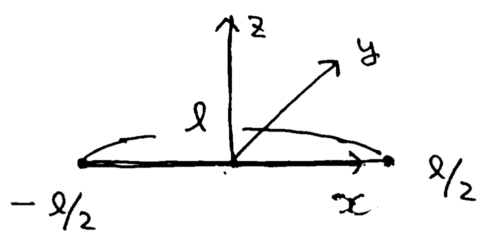
3-4 慣性~~モーメント~~テンソルの具体例

(I) 細長い棒 (長さ ℓ)



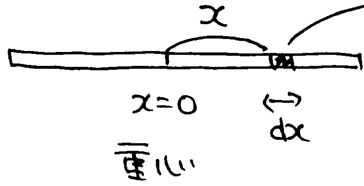
線密度 τ : 一定

全質量 $M = \tau \ell$

重心まわりの慣性モーメント

$$I_{G\,zz} = \sum_i m_i (x_i^2 + y_i^2) \qquad y_i = 0$$

$$= \sum_i m_i x_i^2$$



$m_i = \tau\, dx$

$$I_{G\,zz} = \int_{-\ell/2}^{\ell/2} x^2 \tau\, dx = \tau \times \left[\frac{x^3}{3} \right]_{-\ell/2}^{\ell/2} = \frac{\tau \ell^3}{12}$$

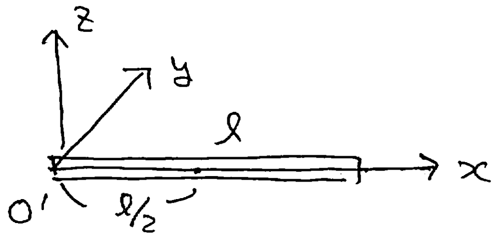
$$= \frac{M}{12} \ell^2$$

同様に

$$I_{G\,yy} = \frac{M}{12} \ell^2$$

$$I_{G\,xx} = 0$$

また $\quad I_{G\,xy} = I_{G\,yz} = I_{G\,zx} = 0$



O' についての慣性テンソル

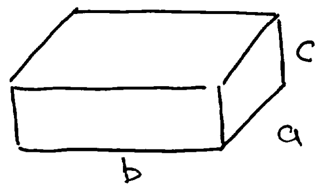
$x_G = \frac{\ell}{2}$, $y_G = z_G = 0$

平行軸の定理より

$$
I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{M}{4}\ell^2 & 0 \\ 0 & 0 & \frac{M}{4}\ell^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{M}{12}\ell^2 & 0 \\ 0 & 0 & \frac{M}{12}\ell^2 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{M}{3}\ell^2 & 0 \\ 0 & 0 & \frac{M}{3}\ell^2 \end{pmatrix}
$$

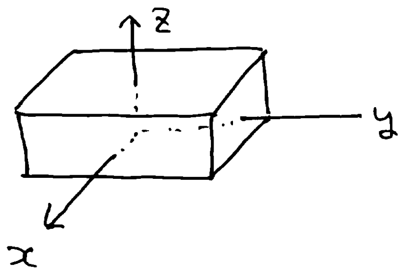
(Ⅱ) 直方体

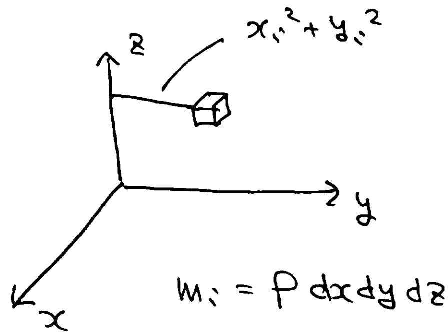


密度 ρ ：一定

全質量 $M = \rho abc$

重心まわりの慣性モーメント





$$I_{G\,zz} = \sum_i m_i (x_i^2 + y_i^2)$$

$$= \int_{\substack{-\frac{a}{2} < x < \frac{a}{2} \\ -\frac{b}{2} < y < \frac{b}{2} \\ -\frac{c}{2} < z < \frac{c}{2}}} (x^2 + y^2)\, \rho\, dx\, dy\, dz$$

$$= \rho \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, (x^2 + y^2) \underbrace{\int_{-c/2}^{c/2} dz}_{c}$$

$$= \rho c \int_{-a/2}^{a/2} dx \left[x^2 y + \frac{1}{3} y^3 \right]_{-b/2}^{b/2}$$

$$I_{G\,zz} = \rho c \int_{-a/2}^{a/2} dx \quad b x^2 + \frac{b^3}{12}$$

$$= \rho c \left[\frac{b}{3} x^3 + \frac{b^3}{12} \right]_{-a/2}^{a/2}$$

$$= \rho c \left(\frac{b}{12} \cdot a^3 + \frac{b^3}{12} a \right)$$

$M = \rho abc$

$$= \frac{M}{12} (a^2 + b^2)$$

同様に $$I_{G\,xx} = \frac{M}{12} (b^2 + c^2)$$

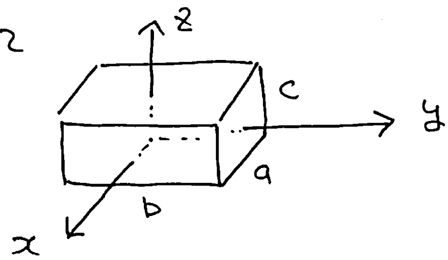
$$I_{G\,yy} = \frac{M}{12} (a^2 + c^2)$$

さらに
$$I_{G\,xy} = - \int_{\substack{-a/2 < x < a/2 \\ -b/2 < y < b/2 \\ -c/2 < z < c/2}} x y \rho \, dx dy dz = 0$$

同様に

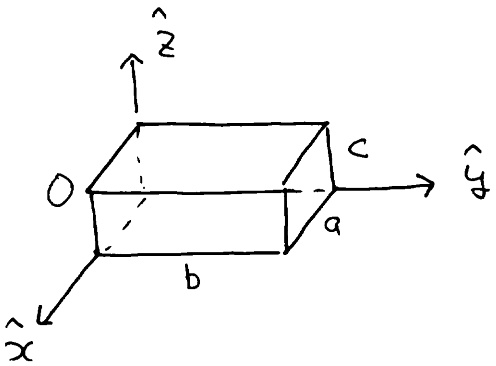
$$I_{G\,yz} = I_{G\,zx} = 0$$

従って



$$M = \rho abc$$

$$I_G = \begin{pmatrix} \frac{M}{12}(b^2+c^2) & 0 & 0 \\ 0 & \frac{M}{12}(a^2+c^2) & 0 \\ 0 & 0 & \frac{M}{12}(a^2+b^2) \end{pmatrix}$$



原点を代表点としてみる.

$$\hat{\mathbb{r}}_G = \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$$

平行軸の定理より

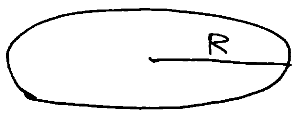
$$\hat{I} = I_M + I_G$$

$$= \begin{pmatrix} \frac{M}{4}(b^2+c^2) & -\frac{M}{4}ab & -\frac{M}{4}ac \\ -\frac{M}{4}ab & \frac{M}{4}(a^2+c^2) & -\frac{M}{4}bc \\ -\frac{M}{4}ac & -\frac{M}{4}bc & \frac{M}{4}(a^2+b^2) \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{M}{12}(b^2+c^2) & 0 & 0 \\ 0 & \frac{M}{12}(a^2+c^2) & 0 \\ 0 & 0 & \frac{M}{12}(a^2+b^2) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{M}{3}(b^2+c^2) & -\frac{M}{4}ab & -\frac{M}{4}ac \\ -\frac{M}{4}ab & \frac{M}{3}(a^2+c^2) & -\frac{M}{4}bc \\ -\frac{M}{4}ac & -\frac{M}{4}bc & \frac{M}{3}(a^2+b^2) \end{pmatrix}$$

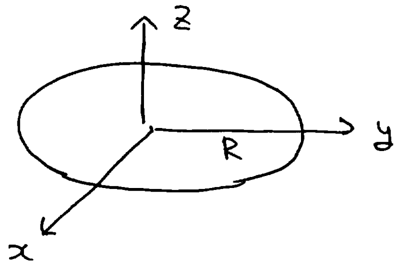
(Ⅲ) 薄い円板

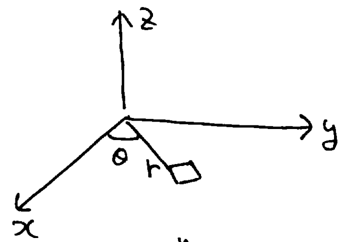


面密度 σ ：一定

全質量 $M = \pi R^2 \sigma$

重心まわりの慣性モーメント





$m_i = \sigma dr \cdot r d\theta$

$$I_{Gzz} = \sum m_i (x_i^2 + y_i^2)$$

$$= \int_{\substack{0 \le r \le R \\ 0 \le \theta < 2\pi}} r^2 \sigma \, r dr d\theta$$

$$= \sigma \underbrace{\int_0^R r^3 dr}_{\frac{R^4}{4}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$= \frac{\pi}{2} R^4 \sigma$$

$M = \pi R^2 \sigma$

$$= \frac{M}{2} R^2$$

$$I_{G\,xx} = \sum_i m_i (y_i^2 + z_i^2) \qquad z_i = 0$$

$$= \sum_i m_i y_i^2$$

$$y = r\sin\theta$$

$$m_i = \sigma\, dr \cdot r\, d\theta$$

$$= \int_{\substack{0 \le r \le R \\ 0 \le \theta < 2\pi}} r^2 \sin^2\theta \; \sigma\, r\, dr\, d\theta$$

$$= \sigma \underbrace{\int_0^R dr\; r^3}_{\frac{R^4}{4}} \underbrace{\int_0^{2\pi} d\theta\; \sin^2\theta}_{\pi}$$

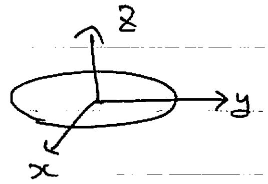
$$= \frac{\pi}{4} R^4 \sigma \qquad M = \pi R^2 \sigma$$

$$= \frac{1}{4} M R^2$$

同様に

$$I_{G\,yy} = \frac{1}{4} M R^2$$

$$I_G = \begin{pmatrix} \frac{M}{4}R^2 & 0 & 0 \\ 0 & \frac{M}{4}R^2 & 0 \\ 0 & 0 & \frac{M}{2}R^2 \end{pmatrix}$$



慣性乗積は すべて ゼロ

$$I_{G\,yz} = -\sum m_i y_i z_i = 0 \qquad z_i = 0$$

$$I_{G\,xz} = -\sum m_i x_i z_i = 0$$

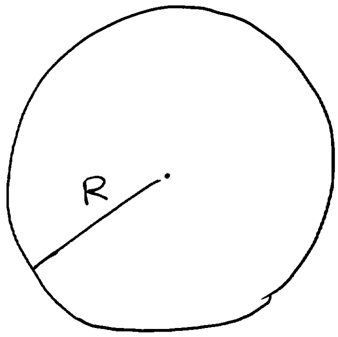
$$I_{G\,xy} = -\sum m_i x_i y_i$$

$$m_i = \sigma\, r\,dr\,d\theta$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$= -\int r^2 \sin\theta\cos\theta \cdot \sigma\, r\,dr\,d\theta$$

$$= -\sigma \int r^3 dr \underbrace{\int_0^{2\pi} \sin\theta\cos\theta\, d\theta}_{=0}$$

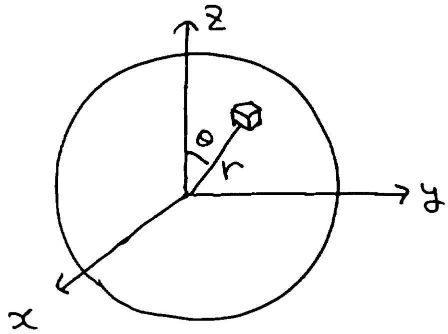
(Ⅳ) 球



密度 ρ : 一定

全質量 $M = \frac{4\pi}{3} R^3 \rho$

重心まわりの慣性モーメント



$m_i \Rightarrow \rho\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$

$x_i^2 + y_i^2 \rightarrow r^2 \sin^2\theta$

$$I_{Gzz} = \sum_i m_i (x_i^2 + y_i^2)$$

$$= \int_{\substack{0 \le r \le R \\ 0 \le \theta < \pi \\ 0 \le \varphi < 2\pi}} r^2 \sin^2\theta \;\; \rho\, r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$= \rho \int_0^R r^4 dr \int_0^\pi \sin^3\theta \, d\theta \int d\varphi$$

$$I_{Gzz} = \rho \cdot \frac{R^5}{5} \cdot \frac{4}{3} \cdot 2\pi$$

$$= \frac{8\pi}{15} R^5 \rho \qquad M = \frac{4\pi}{3} R^3 \rho$$

$$= \frac{2}{5} M R^2$$

同様に

$$I_{Gxx} = I_{Gyy} = \frac{2}{5} M R^2$$

$$\sin^3\theta = \frac{1}{4}(-\sin 3\theta + 3\sin\theta) \quad より$$

$$\int_0^{\pi} \sin^3\theta \, d\theta = \frac{1}{4}\int_0^{\pi} (-\sin 3\theta + 3\sin\theta)\, d\theta$$

$$= \frac{1}{4}\left[\frac{1}{3}\cos 3\theta - 3\cos\theta\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{1}{3}\cos 3\pi - 3\cos\pi - \frac{1}{3} + 3\right]$$

$$= \frac{1}{4}\left[2\left(3 - \frac{1}{3}\right)\right] = \frac{4}{3}$$