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Gravothermal catastrophe and Tsallis' generalized entropy of self-gravitating systems

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Abstract

We present a first physical application of Tsallis' generalized entropy to the thermodynamics of self-gravitating systems. The stellar system confined in a spherical cavity of radius r_e exhibits an instability, so-called gravothermal catastrophe, which has been originally investigated by Antonov (Vestn. Leningrad Gros. Univ. 7 (1962) 135) and Lynden-Bell and Wood (Mon. Not. R. Astron. Soc. 138 (1968) 495) on the basis of the maximum entropy principle for the phase-space distribution function. In contrast to previous analyses using the Boltzmann–Gibbs entropy, we apply the Tsallis-type generalized entropy to seek the equilibrium criteria. Then the distribution function of Vlassov–Poisson system can be reduced to the stellar polytrope system. Evaluating the second variation of Tsallis entropy and solving the zero eigenvalue problem explicitly, we find that the gravothermal instability appears in cases with polytrope index $n > 5$. The critical point characterizing the onset of instability are obtained, which exactly matches with the results derived from the standard turning-point analysis. The results give an important suggestion that the Tsallis' generalized entropy is indeed applicable and viable to the long-range nature of the self-gravitating system. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In any subject of astrophysics and cosmology, many-body gravitating systems play an essential role. In general, dynamics of such systems are quite difficult to understand and the long-range nature of gravitational interaction prevents us from applying the statistical mechanics. Thermodynamics of self-gravitating systems also shows some peculiar features such as a negative specific heat and an absence of global entropy maxima, which greatly differ from usual thermodynamic systems.

To see the peculiarity of self-gravitating systems, consider a system confined within a spherical adiabatic wall. We assume that the particles in this system interact via Newton gravity and bounce elastically from the wall. Keeping the energy and the total mass of the system constant, a thermodynamic description of self-gravitating systems leads to an interesting conclusion. When the central mass density is sufficiently high, no equilibrium state exists and the system can persistently undertake a strong central condensation. This instability is known as *gravothermal catastrophe*, originally investigated by Antonov [1] and Lynden-Bell and Wood [2] (see also Refs. [3–6]). They define the entropy of distribution function in phase-space and seek the equilibrium condition whether the system exhibits the maximum entropy state. In their analyses, they treat the Boltzmann–Gibbs entropy,

$$S_{\rm BG} = -\int f \ln f \, d^3\boldsymbol{x} \, d^3\boldsymbol{p} \,, \tag{1}$$

where $f(\boldsymbol{x},\boldsymbol{p})$ is the distribution function in phase-space.

While the gravothermal catastrophe found in early studies has been widely accepted as a fundamental astrophysical process and plays an essential role in the dynamics of globular clusters [7–10], a naive but natural question arises. Why should we use the Boltzmann–Gibbs entropy when looking for a probable entropy state? The choice of entropy severely restricts the functional form of the distribution function f. In a spherically symmetric system with isotropic velocity distribution, the entropy (1) leads to the isothermal gas distribution. The equilibrium configuration, however, cannot be unique in the self-gravitating systems. There remains a possibility of another choice of entropy in order to determine the most probable state.

Indeed, in the view of statistical mechanics and thermodynamics, it has recently been known that the standard formalism based on the Boltzmann–Gibbs entropy cannot deal with a variety of interesting physical problems and serious difficulties arise when applying the Boltzmann–Gibbs statistical mechanics (e.g., [11–13]). After introducing a family of generalized entropies by Tsallis [14], a new framework of thermodynamic structure has been extensively discussed [15–19] and some physical applications has been presented successfully (e.g., [20–22]; see [23,24] for comprehensive reviews). Although there still remains a fundamental issue on the consistency with the thermodynamic relations [19], the new formalism is expected to be applied in a quite wide area including physics, astronomy, biology, economics, etc. Especially, in many astrophysical problems involving the long-range nature of gravity, the nonextensive generalization of statistical and/or thermodynamic treatment should deserve further consideration.

At present, however, no specific examples have been reported theoretically, except for the experimental result [25] or observational evidence [26].

If one applies Tsallis' generalized entropy to the above problem in self-gravitating systems, the equilibrium condition is explored by using

$$S_q = -\frac{1}{q-1}\int (f^q - f)\, d^3\boldsymbol{x}\, d^3\boldsymbol{p}\,, \tag{2}$$

instead of (1). Here, the parameter q is chosen as $q \neq 1$, as a possible generalization of the Boltzmann–Gibbs entropy. Using (2), an attempt to determine the most probable state has been made by Plastino and Plastino [27,28]. They found that the equilibrium configuration reduces to the polytropic gaseous system, which has been widely utilized in a study of the stellar structure [29,30].

Based on their result, we then pursue to investigate the stability of self-gravitating system confined within a box and try to answer the crucial question; how the equilibrium condition is altered when applying the entropy (2)? More basically, does the Tsallis type entropy correctly predict the equilibrium condition of a stellar polytrope system?

This paper is organized as follows. In Section 2, we briefly review that the most probable state determined by the Tsallis entropy reduces to the stellar polytrope. Then we proceed to the stability analysis based on the standard turning-point analysis in Section 3. In Section 4, stability/instability criterion is rederived from the second variation of entropy. The stability/instability criterion can be obtained by solving zero-eigenvalue problem, which exactly matches the criterion from the standard turning-point analysis. Also, the geometrical construction of stability/instability criterion is discussed in terms of homology invariants. Finally, Section 5 is devoted to conclusions and discussion.

2. Tsallis entropy and stellar polytrope

In this section, just for the notational convenience and the subsequent analyses, we start to review the work by Plastino and Plastino [27] that the extremum state of the Tsallis entropy just reduces to the stellar polytrope in Section 2.1. Adopting their extremum state, the equilibrium configuration is then determined by solving the conditions of hydrostatic equilibrium in Section 2.2. A family of equilibrium sequence referred to as the Emden solutions is obtained and characterized by the homology invariants, which is subsequently used in the stability analysis.

2.1. The principle of maximum entropy

Consider a system containing N particles which are confined within a cavity of hard sphere. The radius of cavity is r_e and each particle is assumed to have the same mass m. Then, all the information of this system can be described by an N-body distribution function $f_N(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N;\boldsymbol{p}_1,\ldots,\boldsymbol{p}_N;t)$, defined in the $6N$-dimensional phase-space. The evolution of such a system is governed by the collisionless Boltzmann equation, which is generally intractable. Instead of using the full distribution, it is better to employ a coarse-grained distribution function whose value at any phase-space point $(\boldsymbol{x},\boldsymbol{p})$ is averaged over some specified volume centered on $(\boldsymbol{x},\boldsymbol{p})$. In this treatment, the correlation between particles is smeared out. Thus, the system can be regarded as a gravitating gaseous system. The phase space distribution is simply described by the "one-particle distribution function" $f(\boldsymbol{x},\boldsymbol{p};t)$, which greatly reduces the problem to a tractable level.

Adopting the mean-field treatment, we now investigate the equilibrium configuration and stability of a gravitating gaseous system with the help of statistical mechanics. In this approach, the equilibrium configuration can be determined by the principle of maximum entropy, which specifies the distribution function f that maximizes the entropy. As is well-known, however, the global maximum does not exist in self-gravitating systems [1,2,8]. Hence, we try to look for the local extrema $\delta S=0$ and seek the criterion whether these extrema are really local maxima or not. Previously, most previous work has extensively discussed this issue on the basis of the Boltzmann–Gibbs entropy (1), in which case the extremum solution reduces to the isothermal gas system. In this paper, with a particular attention to the Tsallis-type generalized entropy (2), we seek the equilibrium configuration under the mass and the energy conservation:

$$M \equiv \int f \, d^3\boldsymbol{x}\, d^3\boldsymbol{v} \,, \tag{3}$$

$$E = K + U \equiv \frac{1}{2}\int v^2 f \, d^3\boldsymbol{x}\, d^3\boldsymbol{v} + \frac{1}{2}\int \Phi(\boldsymbol{x}) f \, d^3\boldsymbol{x}\, d^3\boldsymbol{v} \,, \tag{4}$$

where the quantity Φ denotes gravitational potential:

$$\Phi(\boldsymbol{x}) = -G \int \frac{f(\boldsymbol{y},\boldsymbol{v})}{|\boldsymbol{x}-\boldsymbol{y}|} \, d^3\boldsymbol{y}\, d^3\boldsymbol{v} \,, \tag{5}$$

with $\boldsymbol{v}=\boldsymbol{p}/m$. Here, we adopted the standard definition of mean value for the mass M and the energy E. The remark on the use of other definitions such as q-generalized mean value [17] is discussed in Section 5.

The extremum entropy state can be derived by varying S_q with respect to f. Using the Lagrange multipliers α and β, the extremum solution subject to constraints (3) and (4) is obtained from

$$\delta S_q - \alpha\delta M - \beta\delta E = 0 \,, \tag{6}$$

which leads to

$$\int \left\{ -\frac{1}{q-1}(q f^{q-1} - 1) - \alpha - \beta\left(\frac{1}{2}v^2 + \Phi\right) \right\} \delta f \, d^3\boldsymbol{x}\, d^3\boldsymbol{v} = 0 \,. \tag{7}$$

Here the relation $\int \delta\Phi f \, d^3\boldsymbol{x}\, d^3\boldsymbol{v} = \int \Phi\, \delta f \, d^3\boldsymbol{x}\, d^3\boldsymbol{v}$ is used in deriving the above expression. Since constraint (6) must be satisfied independently of the choice of δf, we obtain

$$\frac{1}{q-1}(q f^{q-1} - 1) + \alpha + \beta\left(\frac{1}{2}v^2 + \Phi\right) = 0 \,, \tag{8}$$

which reduces to the following distribution function:

$$f(\boldsymbol{x},\boldsymbol{v}) = A[\Phi_0 - \Phi(\boldsymbol{x}) - \tfrac{1}{2}v^2]^{1/(q-1)}\,, \tag{9}$$

where we define the constants A and Φ_0:

$$A = \left\{\left(\frac{q-1}{q}\right)\beta\right\}^{1/(q-1)}, \qquad \Phi_0 = \frac{1-(q-1)\alpha}{(q-1)\beta}\,. \tag{10}$$

The functional form of the distribution function (9) implies that the extremum solution is indeed equivalent to the polytropic gaseous systems [8,27]. The density profile $\rho(r)$ at radius $r = |\boldsymbol{x}|$ can be expressed using (9):

$$\begin{aligned}\rho(\boldsymbol{x}) &\equiv \int f\,\mathrm{d}^3\boldsymbol{v},\\ &= 4\sqrt{2}\pi B\left(\frac{3}{2},\frac{q}{q-1}\right)A\{\Phi_0 - \Phi(\boldsymbol{x})\}^{1/(q-1)+3/2}\end{aligned} \tag{11}$$

with $B(a,b)$ being the β function. On the other hand, in the case of isotropic velocity distribution, the pressure becomes

$$\begin{aligned}P(\boldsymbol{x}) &\equiv \int \frac{1}{3}v^2 f\,\mathrm{d}^3\boldsymbol{v},\\ &= \left(\frac{1}{q-1}+\frac{5}{2}\right)^{-1}\rho(\boldsymbol{x})\{\Phi_0 - \Phi(\boldsymbol{x})\}\,.\end{aligned} \tag{12}$$

Thus, these two equations lead to the relation

$$P \propto \rho^{(5q-3)/(3q-1)}\,, \tag{13}$$

which corresponds to the polytropic equation of state, $P \propto \rho^{1+1/n}$, and the polytrope index n is connected with Tsallis' q-parameter as follows:

$$n = \frac{1}{q-1} + \frac{3}{2}\,. \tag{14}$$

Note that the polytrope index n should be positive in any astrophysical system. It has also been argued that the values of n in the interval $0 < n < \frac{3}{2}$ are unphysical [8], which restricts the parameter q to be larger than unity. Further, a simple argument shows that the locally maximum entropy can only be attained in the spherically symmetric configuration [1,2]. Although the extremum solution (9) does not restrict any symmetry and any value of the parameter q, we hereafter restrict our attention to the spherically symmetric case with $q \geqslant 1$.

The resultant distribution (9) and relation (14) further imply that the polytrope gas is equivalent to the isothermal gaseous sphere in the limit $q \to 1$ or $n \to +\infty$, which is in fact obtained when adopting the Boltzmann–Gibbs entropy (1). To check this consistency, let us take the limit $q \to 1$. Introducing the new constant T:

$$T \equiv (q-1)\left\{4\sqrt{2}\pi B\left(\frac{3}{2},\frac{q}{q-1}\right)A\right\}^{-(2q-2)/(3q-1)}, \tag{15}$$

one can rewrite the distribution function (9) using relations (14) and (15) as

$$f(r,v) = \frac{1}{4\sqrt{2}\pi B(3/2, n-1/2)} \times \frac{\rho(r)}{\{(n-3/2)T\rho^{1/n}(r)\}^{3/2}}\left[1-\frac{v^2/2}{(n-3/2)T\rho^{1/n}(r)}\right]^{n-3/2}. \tag{16}$$

Also, the polytropic equation of state becomes

$$P(r) = \left(\frac{n-3/2}{n+1}\right) T\rho^{1+1/n}(r). \tag{17}$$

Then using the fact that $B(\frac{3}{2}, n-\frac{1}{2}) \to (\pi/4n^3)^{1/2}$ in the limit $n \to \infty$, the distribution function asymptotically approaches

$$f(r,v) \overset{n\to+\infty}{\longrightarrow} \frac{1}{(2\pi T)^{3/2}}\,\rho(r)\,\mathrm{e}^{-v^2/2T} \tag{18}$$

and the equation of state reduces to that of the isothermal gas, $P = \rho T$. Expression (18) shows that the velocity distribution is indeed Maxwellian and functional form of the distribution is fully specified by the 'temperature', T (velocity dispersion).

2.2. Emden solution

While the extremum state is determined by the variation of entropy, the distribution function (9) does not yet completely specify the equilibrium configuration. In expression (9), the gravitational potential Φ appears, which implicitly involves f again (see Eq. (5)). To break this roundabout, an explicit expression for Φ or ρ is needed.

In the spherically symmetric configuration, the gravitational potential given by (5) satisfies the following Poisson equation:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r}\right) = 4\pi G\rho(r). \tag{19}$$

Combining (19) with (11), we obtain the ordinary differential equation for Φ. Alternatively, a set of equations are derived using (11), (12) and (19):

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{Gm(r)}{r^2}\rho(r), \tag{20}$$

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi\rho(r)r^2, \tag{21}$$

which represent the hydrostatic equilibrium. The quantity $m(r)$ denotes the mass evaluated at the radius r inside the wall. We then introduce the dimensionless quantities:

$$\rho = \rho_c[\theta(\xi)]^n, \qquad r = \left\{ \frac{(n+1)P_c}{4\pi G\rho_c^2} \right\}^{1/2} \xi \,, \tag{22}$$

which yields the following ordinary differential equation:

$$\theta'' + \frac{2}{\xi}\theta' + \theta^n = 0 \,, \tag{23}$$

where prime denotes the derivative with respect to ξ. The quantities ρ_c and P_c in (22) are the density and the pressure at $r=0$, respectively. To obtain the physically relevant solutions, we solve Eq. (23) with the following boundary condition:

$$\theta(0) = 1, \qquad \theta'(0) = 0 \,. \tag{24}$$

A family of solutions satisfying (24) is referred to as the *Emden solution*, which is well-known in the subject of stellar structure and details of the solutions can be found in standard textbooks (e.g., Ref. [29, Chapter IV]). Except for the few cases $n = 0$, 1 and 5, the Emden solution cannot be expressed in terms of the elementary functions. So the solution with general index n is obtained numerically.

Notice the fact that Eq. (23) is invariant under the transformation $\xi \to A\xi$, $\theta \to A^{-2/(n-1)}\theta$, where A is an arbitrary constant. This implies that, given a solution with some value of $\theta(0)$, we can obtain the solution with any other value of $\theta(0)$ by simple rescaling. Therefore, only one of the two integration constants in (23) is actually nontrivial. This fact allows us to reduce the degree of equation from two to one by suitable choice of variables. One such set of variables is

$$u \equiv \frac{4\pi r^3 \rho(r)}{m(r)} = -\frac{\xi\theta^n}{\theta'} \,, \tag{25}$$

$$v \equiv \frac{\rho(r)}{P(r)} \frac{Gm(r)}{r} = -(n+1)\frac{\xi\theta'}{\theta} \tag{26}$$

which are called *homology invariants* [29,30]. That is, for a fixed n, all the solutions depicted in (u, v)-plane lie on the same trajectory. In terms of these variables, Eq. (23) can be written as

$$\frac{u}{v}\frac{dv}{du} = \frac{(n+1)(u-1)+v}{(n+1)(3-u)-nv} \,. \tag{27}$$

Fig. 1 shows the Emden solutions for various polytrope indices. As indicated by the boundary condition, (24), all the trajectory start from $(u, v) = (3, 0)$, which represents the center of the configuration, $r=0$. As the radius r increases, the trajectory monotonically moves to the upper-left direction in (u, v)-plane, as marked by arrow. For larger radius, the solution with polytrope index $n > 5$ spirals around a fixed point, while the trajectories with $n < 5$ continue to approach the point $(u, v)=(0, \infty)$. The marginal case is $n=5$. The trajectory monotonically changes and asymptotically reaches $(u, v)=(0, 6)$. These differences can be explained by the behavior of outer envelope $\rho(r)$ as follows.

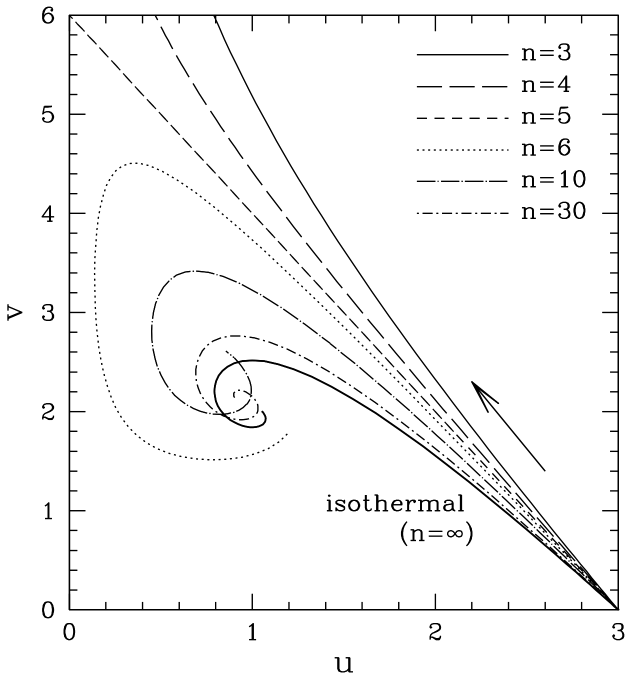

Fig. 1. Trajectories of Emden solution in (u, v)-plane.

In cases of $n < 5$, the density $\rho(r)$ falls off rapidly and it eventually vanishes at a finite size, where the pressure also becomes zero. Since the mass is finite and the polytropic relation $P \propto \rho^{1+1/n}$ holds, the quantities (u, v) becomes $(u, v) = (0, \infty)$ from (25) and (26). On the other hand, when $n > 5$, the outer profile of the density falls off less steeply than $\rho \propto r^{-5}$ [8]. In this case, the trajectory in (u, v)-plane cannot reach $u = 0$. It must be bounded within the interval $0 < u < 3$ and shows the oscillatory behavior, as depicted in Fig. 1.

The characteristic feature seen in Fig. 1 provides us an important suggestion about stability of the polytrope gas sphere. This point will be clarified by the turning-point analysis and the second variation of entropy in the subsequent section.

3. Stability/instability criterion from the turning-point analysis

We are specifically concerned with the stability/instability of static equilibria from the thermodynamic point of view. For this purpose, following the discussion in [2,5,6], we first apply the standard-turning point analysis to the equilibrium sequence obtained in the previous section.

Recall that the system is confined within the spherical adiabatic wall of radius r_e. For fixing polytrope index n, the local extremum satisfying $\delta S_q = 0$ is characterized by the energy E, mass M and radius of rigid sphere r_e. In the absence of instability, any value of $E\,(-\infty < E < \infty)$, $M\,(0 < M < \infty)$ and $r_e(0 < r_e < \infty)$ are accommodated by a suitable choice of ρ_c and P_c, but, there exists a lower bound on the dimensionless quantity Er_e/GM^2 for the polytrope gas solutions. To show this, we compute the total energy contained within the hard sphere of a radius r_e. The kinetic and potential energies, K and U are, respectively, expressed as

$$K = \frac{3}{2}\int_0^{r_e} \mathrm{d}r\, 4\pi r^2 P(r) \tag{28}$$

and

$$U = -\int_0^{r_e} \mathrm{d}r \frac{Gm}{r}\frac{\mathrm{d}m}{\mathrm{d}r} = -\frac{G}{2}\int_0^{r_e} \frac{\mathrm{d}r}{r}\frac{\mathrm{d}}{\mathrm{d}r}(m^2) = -\frac{GM^2}{2r_e} - \frac{1}{2}\int_0^{r_e} \mathrm{d}r \frac{Gm^2}{r^2}$$
$$= 4\pi r_e^3 P_e - 3\int_0^{r_e} \mathrm{d}r\, 4\pi r^2 P(r)\,. \tag{29}$$

Combining these results and using (A.3) in Appendix A, the total energy E can be written in a compact form:

$$E = K + U = 4\pi r_e^3 P_e - \frac{3}{2}\int_0^{r_e} \mathrm{d}r\, 4\pi r^2 P(r)$$
$$= \frac{1}{n-5}\left[\frac{3}{2}\left\{\frac{GM^2}{r_e} - (n+1)\frac{MP_e}{\rho_e}\right\} + (n-2)4\pi r_e^3 P_e\right]\,, \tag{30}$$

where the subscript ($_e$) means a quantity evaluated at the boundary r_e. Then, the dimensionless quantity λ is introduced and expressed in terms of homology invariants at the boundary $r = r_e$:

$$\lambda \equiv -\frac{Er_e}{GM^2} = -\frac{1}{n-5}\left[\frac{3}{2}\left\{1 - (n+1)\frac{1}{v_e}\right\} + (n-2)\frac{u_e}{v_e}\right]\,. \tag{31}$$

Fig. 2 shows the quantity λ as a function of the ratio of the central density to that at the boundary, ρ_c/ρ_e. In each panel, lines indicate a series of local extrema with a different polytrope index n. Note that all solutions satisfying $\delta S_q = 0$ must lie on this curve.

At a first glance of Fig. 2, we notice that λ-curves are bounded from above and have peaks in the case of $n > 5$ (*right panel*). We call these peaks critical points. On the other hand, curves for $n \leqslant 5$ monotonically increase (*left panel*). It follows that, in the case of $n > 5$, several extremum states correspond to a single value of λ in some range of λ. Thus we can deduce the existence of unstable state for $n > 5$ as follows. Suppose that a polytropic gaseous sphere with small radius $r_e \to 0$, $(\rho_c/\rho_e \to 1)$ is stable, i.e., $\delta^2 S_q < 0$ (see Appendix C for rigorous proof). Due to the continuity, entropy S_q must be local maximum along each λ-curve. This is true as long as λ increases monotonically. When the ratio ρ_c/ρ_e exceeds the critical point, however, the extremum configuration cannot be stable. If this is stable, the entropy has to be a local maximum there, which implies that there exists a local minimum between these two local maxima. This contradicts with the fact that all extremum state must lie on

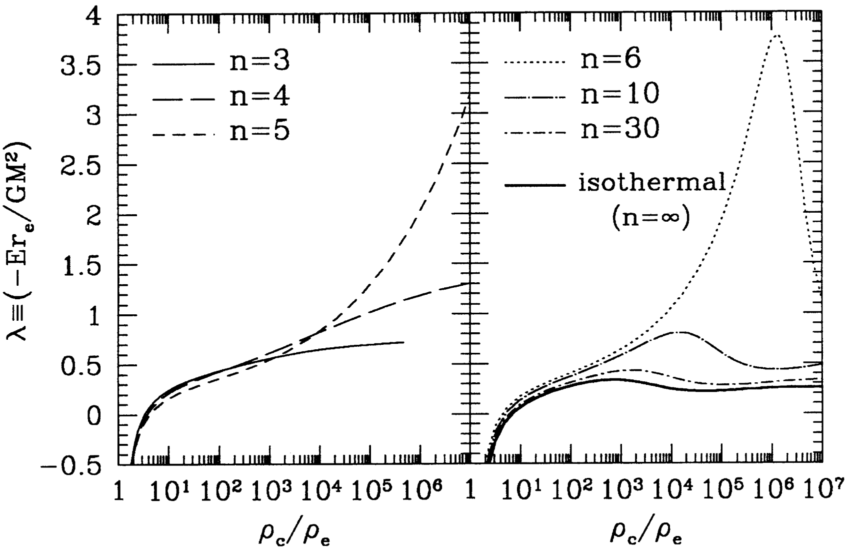

Fig. 2. Energy–radius–density contrast relationship.

the λ-curve. Therefore, local maximum of the entropy can only be attained at the density ρ_c/ρ_e below the critical point and the stability/instability criterion is thus obtained from the condition that the quantity

$$\frac{\mathrm{d}\lambda}{\mathrm{d}(\rho_c/\rho_e)} = 0 \tag{32}$$

first vanishes as the ratio ρ_c/ρ_e increases. From the monotonicity of $\rho(r)$, this condition implies $\mathrm{d}\lambda/\mathrm{d}\xi_e = 0$. Therefore, using the dimensionless quantities (22) and Eq. (23), the stability/instability criterion reduces to

$$0 = \frac{\mathrm{d}\lambda}{\mathrm{d}\xi_e} \equiv \frac{n-2}{n-5}\,\frac{g(u_e, v_e)}{2v_e\,\xi_e}\,, \tag{33}$$

where

$$g(u_e, v_e) = 4u_e^2 + 2u_e v_e - \left\{8 + 3\left(\frac{n+1}{n-2}\right)\right\} u_e - \frac{3}{n-2} v_e + 3\left(\frac{n+1}{n-2}\right)\,. \tag{34}$$

As will be shown in the next section, especially in Eq. (53), this criterion can be derived also by means of an explicit method, i.e., evaluation of the second-order variation of the entropy.

In Table 1, numerical values of dimensionless quantities λ and ρ_c/ρ_e evaluated at the critical point are summarized. Table 1 shows that both two quantities decrease as n increases. In the limit $n \to \infty$ (or $q \to 1$), they asymptotically approach the well-known results of the isothermal sphere [1–3,5].

Table 1
Energy–radius–mass relation and density contrast between center and edge evaluated at a critical point for given polytrope index n or q

n	q	$\lambda(=-\frac{Er_e}{GM_e^2})$	$\frac{\rho_c}{\rho_e}$
5	$\frac{9}{7}$	—	—
6	1.22	3.78	1.27×10^6
7	1.18	1.69	1.46×10^5
8	1.15	1.162	4.83×10^4
9	1.13	0.932	2.40×10^4
10	1.12	0.804	1.46×10^4
30	1.04	0.429	1590
50	1.02	0.388	1130
100	1.01	0.360	887
∞	1	0.335	709

4. Stability criterion from the second variation of entropy

In this section, we reconsider the stability/instability of static self-gravitating system by evaluating the second variation of entropy. Then, criterion (33) is rederived, independently of the turning-point analysis.

According to the principle of maximum entropy, the equilibrium state can be attained only when the second variation of entropy $\delta^2 S_q$ around the extremum solution becomes negative, $\delta^2 S_q < 0$. Conversely, the solution becomes unstable if one obtains $\delta^2 S_q > 0$. The condition $\delta^2 S_q = 0$ corresponds to the neutral case, in which the equilibrium configuration becomes neither stable nor unstable. Thus, the stability/instability criterion can be extracted from $\delta^2 S_q = 0$.

Consider the variation of entropy around the extremum configuration, fixing the energy E and the mass M. We deal with the density perturbation $\delta\rho$ around the polytropic gaseous state $\rho(r)$. To be specific, we focus on the radial mode of the density perturbations under the condition:

$$\int_0^{r_e} \mathrm{d}r\, 4\pi r^2 \delta\rho(r) = 0\,, \tag{35}$$

so as to satisfy the mass conservation, $\delta M = 0$. In Appendix B, variation of entropy around the equilibrium configuration is computed up to the second order. The resultant expression for $\delta^2 S_q$ then becomes

$$\delta^2 S_q = -\frac{\tilde{B}_n}{T^{(3/2)/(n-3/2)}}\frac{n+1}{n-3/2}\left[\int \mathrm{d}^3\boldsymbol{x}\left\{\frac{\delta\rho\delta\Phi}{2T} + \frac{n-3/2}{2n}\frac{(\delta\rho)^2}{\rho^{1-1/n}}\right\}\right.$$
$$\left. + \frac{n(n+1)}{(n-3/2)^2}\frac{1}{3T^2W}\left\{\int \mathrm{d}^3\boldsymbol{x}\left(\Phi + \frac{3}{2}\frac{n-3/2}{n}T\rho^{1/n}\right)\delta\rho\right\}^2\right], \tag{36}$$

where the variable $\delta\Phi$ denotes the potential related to the perturbation $\delta\rho$:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\delta\Phi}{\mathrm{d}r}\right)=4\pi G\,\delta\rho(r) \tag{37}$$

and the quantities $\tilde{B}_n$ and W are, respectively, given by

$$\tilde{B}_n=\frac{B(3/2,n+1/2)}{\{B(3/2,n-1/2)\}^{(n-1/2)/(n-3/2)}}\left[4\sqrt{2}\pi\left(n-\frac{3}{2}\right)^{3/2}\right]^{-1/(n-3/2)},$$

$$W=\int \mathrm{d}^3\boldsymbol{x}\,\rho^{1+1/n},$$

which reduce to $\tilde{B}_n\to 1$ and $W\to M$, in the limit of $n\to\infty$(or $q\to 1$).

Based on the result (36), we now discuss the existence or the absence of perturbation mode satisfying $\delta^2 S_q=0$ for given background $\rho(r)$. To simplify the analysis, it is convenient to introduce the new variable $Q(r)$ defined by

$$\delta\rho(r)=\frac{1}{4\pi r^2}\frac{\mathrm{d}Q(r)}{\mathrm{d}r}. \tag{38}$$

Then the mass conservation (35) implies the following boundary condition:

$$Q(0)=Q(r_e)=0. \tag{39}$$

In terms of $Q(r)$, Eq. (36) becomes

$$\delta^2 S_q=\frac{\tilde{B}_n}{T^{(3/2)/(n-3/2)}}\frac{n+1}{n-3/2}A[Q], \tag{40}$$

where

$$A[Q]\equiv-\int_0^{r_e}\mathrm{d}r\left\{\frac{\delta\Phi}{2T}\frac{\mathrm{d}Q}{\mathrm{d}r}+\frac{n-3/2}{n}\frac{1}{8\pi r^2\rho^{1-1/n}}\left(\frac{\mathrm{d}Q}{\mathrm{d}r}\right)^2\right\}$$
$$-\frac{1}{3T^2W}\frac{n(n+1)}{(n-3/2)^2}\left[\int_0^{r_e}\mathrm{d}r\left\{\Phi+\frac{3}{2}\left(\frac{n-3/2}{n}\right)T\rho^{1/n}\right\}\frac{\mathrm{d}Q}{\mathrm{d}r}\right]^2. \tag{41}$$

Since both $\tilde{B}_n$ and T are positive, the condition $\delta^2 S_q=0$ is translated to $A[Q]=0$. Thus, apart from the pre-factor, we pay attention to the term $A[Q]$. Integrating (41) by parts and using (37), we obtain

$$A[Q]=\frac{1}{2}\int_0^{r_e}\mathrm{d}r\,Q(r)\left\{\frac{G}{Tr^2}+\frac{n-3/2}{n}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{4\pi r^2\rho^{1-1/n}}\frac{\mathrm{d}}{\mathrm{d}r}\right)\right\}Q(r)$$
$$-\frac{1}{3T^2W}\frac{n(n+1)}{(n-3/2)^2}\left[\int_0^{r_e}\mathrm{d}r\left\{\frac{\mathrm{d}\Phi}{\mathrm{d}r}+\frac{3}{2}\left(\frac{n-3/2}{n^2}\right)T\rho^{-1+1/n}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right\}Q\right]^2$$
$$\equiv-\int_0^{r_e}\mathrm{d}r_1\int_0^{r_e}\mathrm{d}r_2\,Q(r_1)\hat{K}(r_1,r_2)Q(r_2), \tag{42}$$

where the kernel $\hat{K}$ is given by

$$\hat{K}(r_1,r_2) = -\frac{1}{2}\delta_D(r_1-r_2)\left\{\frac{n-3/2}{n}\frac{\mathrm{d}}{\mathrm{d}r_1}\left(\frac{1}{4\pi r_1^2\{\rho(r_1)\}^{1-1/n}}\frac{\mathrm{d}}{\mathrm{d}r_1}\right)+\frac{G}{Tr_1^2}\right\}$$
$$+\frac{1}{3T^2W}\frac{n(n+1)}{(n-3/2)^2}F(r_1)F(r_2) \tag{43}$$

with quantity $F(r)$ being

$$F(r) = \frac{\mathrm{d}\Phi}{\mathrm{d}r}+\frac{3}{2}\left(\frac{n-3/2}{n^2}\right)T\rho^{-1+1/n}\frac{\mathrm{d}\rho}{\mathrm{d}r}\,. \tag{44}$$

Therefore, stability of the background configuration $\rho(r)$ is deduced from the following eigenvalue equation:

$$\int_0^{r_e}\mathrm{d}r'\,\hat{K}(r,r')Q(r') = \kappa Q(r)\,. \tag{45}$$

From (45), the stable condition $\delta^2 S_q < 0$ indicates that minimum eigenvalue κ_{min} should be positive, while the local minimum of the entropy $\delta^2 S_q > 0$ means $\kappa_{\mathrm{min}} < 0$. Hence, the boundary between stable and the unstable configuration corresponds to the condition $\kappa_{\mathrm{min}} = 0$. Notice that for a given polytrope index n, the minimum eigenvalue κ_{min} of the system with (E,M) depends on the radius of the rigid sphere r_e. For sufficiently small r_e, it can be shown that the configuration should be stable and we obtain the positive eigenvalue, $\kappa_{\mathrm{min}} > 0$ (see Appendix C). Therefore, to seek the boundary of the stability, it is sufficient to investigate the condition $\kappa_{\mathrm{min}}=0$ when increasing r_e. That is, the stability/instability criterion can be extracted from the following zero-eigenvalue equation:

$$\int_0^{r_e}\mathrm{d}r'\,\hat{K}(r,r')Q(r') = 0\,,$$

which yields

$$\hat{L}Q(r) \equiv \left[\frac{\mathrm{d}}{\mathrm{d}r}\left\{\frac{1}{4\pi r^2\rho}\left(\frac{P}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}r}\right\}+\frac{n}{n+1}\frac{G}{r^2}\right]Q(r)$$
$$= \frac{2}{3}\frac{n-3/2}{n+1}\frac{1}{\int_0^{r_e}\mathrm{d}r'\,4\pi r^2 P(r')}\frac{Gm(r)}{r^2}\int_0^{r_e}\mathrm{d}r'\frac{Gm(r')}{r'^2}Q(r')\,. \tag{46}$$

In deriving the above expression, we have utilized relation (17) and the hydrostatic equilibrium conditions, (20) and (21).

The zero-eigenvalue equation (46) is integro-differential equation which seems intractable at a first glance, however, the solution satisfying boundary condition (39) is luckily obtained from the knowledge of background configuration, $\rho(r)$ or $\Phi(r)$. To construct the solution $Q(r)$, first note the action of the operator $\hat{L}$ on $4\pi r^3\rho$ and $m(r)$:

$$\hat{L}(4\pi r^3\rho) = \frac{\mathrm{d}}{\mathrm{d}r}\left\{-\frac{n}{n+1}\frac{Gm(r)}{r}+3\,\frac{P}{\rho}\right\}+\frac{n}{n+1}4\pi G r\rho = \frac{n-3}{n+1}\frac{Gm(r)}{r^2}$$

and

$$\hat{L}m(r) = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{P}{\rho}\right) + \frac{n}{n+1}\frac{Gm(r)}{r^2} = \frac{n-1}{n+1}\frac{Gm(r)}{r^2},$$

where we have used the following relations:

$$\frac{P}{\rho}\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} = -\frac{n}{n+1}\frac{Gm}{r}, \qquad \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{P}{\rho}\right) = -\frac{1}{n+1}\frac{Gm}{r^2}. \tag{47}$$

From the indication of these equations, we put the ansatz of the solution,

$$Q(r) = c_1 4\pi r^3 \rho(r) + c_2 m(r) \tag{48}$$

and determine the coefficients c_1 and c_2 by substituting (48) into (46). We then have

$$\frac{n-3}{n+1}c_1 + \frac{n-1}{n+1}c_2 = \frac{2}{3}\frac{n-3/2}{n+1}\frac{\int_0^{r_e}\mathrm{d}r'(Gm(r')/r'^2)Q(r')}{\int_0^{r_e}\mathrm{d}r'\,4\pi r'^2 P(r')} \equiv \Lambda. \tag{49}$$

Further, recall that the ansatz (48) must satisfy the boundary condition (39). Since the condition $Q(0) = 0$ is automatically satisfied, the remaining condition $Q(r_e) = 0$ requires

$$4\pi r_e^3 \rho_e c_1 + M c_2 = 0. \tag{50}$$

Eqs. (49) and (50) specify the coefficients, which can be expressed in terms of the homology invariants (see definitions (25), (26)):

$$c_1 = \frac{(n+1)\Lambda}{n-3-(n-1)u_e}, \qquad c_2 = -\frac{(n+1)u_e\Lambda}{n-3-(n-1)u_e}. \tag{51}$$

The solution (48) with (51) seems still uncertain because of the quantity Λ in coefficients, which implicitly depends on the solution (48) itself. This means that the value of the coefficients c_1 and c_2 should be further constrained so that the local expression (48) indeed satisfies the nonlocal equations. In other words, the consistency between (48) and Λ puts the condition for the background quantities, ρ_e, P_e and M evaluated at $r = r_e$. This is just the stability/instability criterion we wish to clarify.

Now, substituting the solution (48), we evaluate the quantity Λ explicitly.

$$\Lambda = \frac{2}{3}\frac{n-3/2}{n+1}\frac{c_1\int_0^{r_e}\mathrm{d}r(Gm/r^2)4\pi r^3\rho + c_2\int_0^{r_e}\mathrm{d}r\,Gm^2/r^2}{\int_0^{r_e}\mathrm{d}r\,4\pi r^2 P}. \tag{52}$$

The integrals in the right-hand side of (52) can be rewritten repeating the integration by parts as shown in Appendix A. Together with the coefficients (51), substitution of (A.1)–(A.3) into (52) leads to the stability/instability criterion. In terms of u–v variables, this gives

$$1=\frac{2}{3}\frac{n-3/2}{n-3-(n-1)u_e}\left[\frac{(n-5)u_e}{2u_e+v_e-n-1}+3\right.$$
$$\left.-u_e\left\{\frac{(n-5)\{2u_e+v_e\}}{2u_e+v_e-n-1}+6\right\}\right].$$

After some algebra, the above equation finally reduces to the following quadratic form:

$$0=4u_e^2+2u_ev_e-\left\{8+3\left(\frac{n+1}{n-2}\right)\right\}u_e-\frac{3}{n-2}v_e+3\left(\frac{n+1}{n-2}\right)$$
$$=g(u_e,v_e)\,, \tag{53}$$

which exactly matches the criterion derived from the turning-point analysis (see Eqs. (33) and (34)).

Eq. (53) is the main result of our analysis. It determines the critical point in the (u,v)-plane, where the local extremum entropy is neither maximum nor minimum [5].

To see the geometrical meaning of this criterion explicitly, we translate the result (53) into the constraint in the (u,v)-plane. In the previous section, we see that a family of equilibrium sequences is characterized by the parameters, r_e, E and M. These parameters specify the value of λ defined in (31). In other words, such a polytrope system must lie on the straight line:

$$v=-\frac{n+1}{(n-5)\lambda-3/2}\left[\frac{n-2}{n-1}u-\frac{3}{2}\right] \tag{54}$$

from Eq. (31). Note also that the polytrope solution must lie on the u–v trajectory as shown in Fig. 1. Hence, the equilibrium state with fixed parameter set (r_e,E,M) can exist only if the u–v curve intersects the straight line (54). From the configuration of the u–v trajectory, we notice that the quantity λ is bounded from above, $\lambda\leqslant\lambda_c$ in the case of $n>5$. That is, for the lines (54) with λ larger than λ_c, the intersection ceases to exist. Since the critical value λ_c must satisfy $d\lambda/dr=0$, this yields (33) or (53) and the condition $g(u_e,v_e)=0$ determines the critical point along each u–v trajectory.

In Fig. 3, with a great advantage of u,v-variables, the stability/instability criterion (53) is examined. In each panel, the thick-solid lines denote the trajectories of the Emden solution. The thin-solid lines correspond to the equation $g(u,v)=0$. From Fig. 3, one can observe that the curve $g(u,v)=0$ drastically changes its behavior around the index $n=5$ (compare $n=4.8$ with $n=5.2$ case). Note also the different asymptotic behavior of the Emden solutions between the $n>5$ cases and the $n<5$ cases (see also Fig. 1). As a consequence, the critical point appears when the polytrope index $n>5$ and it disappears in the $n<5$ cases.

For the $n>5$ cases, we also plot the straight line (54) with the critical value λ_c (dashed line). It should be emphasized that three lines, (i.e. the trajectory of the Emden solution, the stability/instability criterion (53) and the condition (54) with the critical value λ_c) intersect at the critical point. For comparison, the result of the isothermal gas

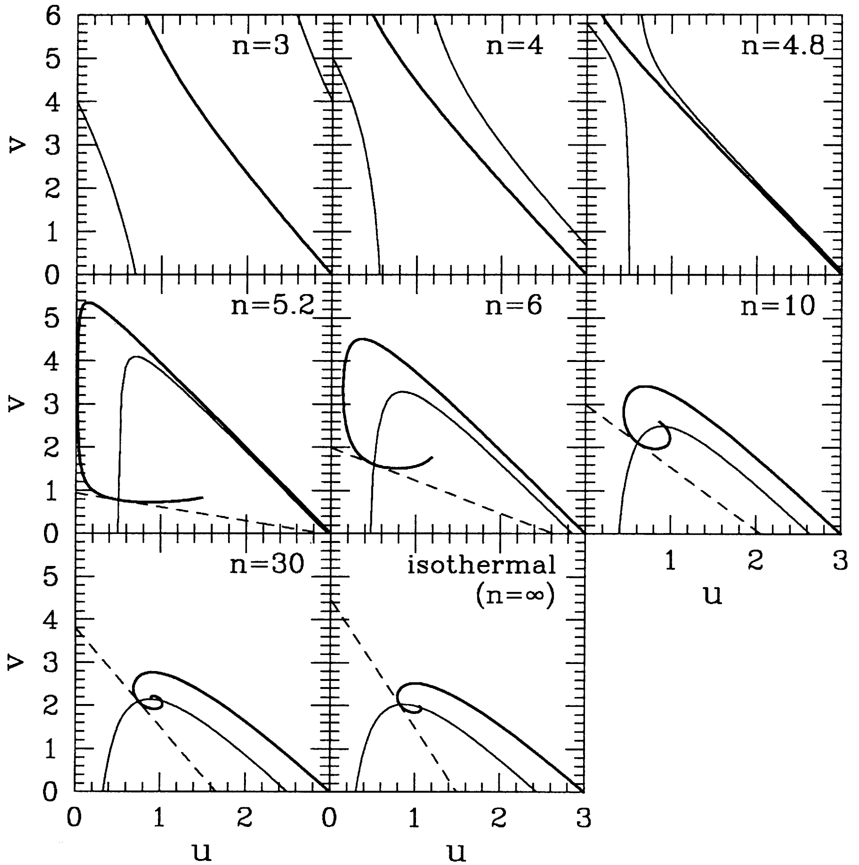

Fig. 3. Stability/instability criterion in (u, v)-plane.

sphere is plotted in Fig. 3. The existence of the critical point still holds in the limit $n \to \infty$ [1–3,5].

5. Discussion and conclusions

In this paper, we have applied Tsallis-type generalized entropy S_q to the problem of seeking the stable distribution of self-gravitating systems. In contrast to previous work using the Boltzmann–Gibbs entropy, the local extremum state of Tsallis-type entropy has been found to be equivalent to the stellar polytrope system and Tsallis' q-parameter is related to the polytrope index n (see Eq. (14)). Then we apply the usual turning-point analysis to explore stability/instability of the local extremum state. We note that the homology invariants (u, v) are very useful to investigate the nature of equilibrium state of the self-gravitating system, including the stability/instability criterion (33). Then we develop the second variation of entropy and reexamine the stability/instability of the local extremum state. The stability/instability criterion (53), which exactly coincides with result (33), can be obtained by solving the integro-differential equation of zero-eigenvalue state.

The results imply the important conclusion that the polytropic gaseous sphere within radius r_e exhibits the gravothermal instability in the case of polytrope index $n > 5$. The characteristic values λ and ρ_c/ρ_e are also evaluated at critical point and presented in Table 1. These values asymptotically approach the well-known result of the isothermal sphere in the limit $n \to \infty$ (or $q \to 1$). In the isothermal case, the gravothermal instability can be interpreted in terms of a negative specific heat which seems to be a common feature in the self-gravitating systems [2]. We note, however, that the system should have sufficient amount of outer normal part in order to trigger the instability; the specific heat for total system should be positive for onset of the instability, although its central part has negative specific heat. A similar argument holds for the polytropic system so that the instability appears for the system of $n > 5$ which can have an elongated outer part. Evaluation of the specific heat based on the canonical ensemble corresponding to Tsallis' type entropy seems to be very important on this issue.

Finally, we comment on the role of constraints M and E to the extremum solution (9). In our present analysis, standard definition of mean values is adopted for mass and energy (see Eqs. (3) and (4)). Tsallis et al. [17] recently showed that this choice yields undesirable divergence in some physical systems including Lévy random walk. To overcome the mathematical difficulty, they suggest that the q-generalized mean value should be used with a correct normalization instead of the standard mean value (see also [18]). According to their suggestion, in Ref. [28], q-generalized, but *un-normalized* mean value was used to define E and M. In this case, the resultant form of extremum state essentially remains unchanged and the polytropic equation of state still holds. We note, however, that if we use the *normalized* q-expectation value correctly, the problematic difficulty conversely arises. This is due to the fact that the potential energy U associated with the distribution function $f(\boldsymbol{x}, \boldsymbol{v})$ (see Eqs. (4) and (5)) becomes nonlinear function of f, while the new formalism in Refs. [17,18] implicitly assumes the linear function. Certainly, the normalized q-expectation value will play an essential role to avoid the unexpected singular behavior, however, no undesirable divergence has appeared in our present analysis. Furthermore, it has been shown that the Tsallis formalism with standard linear mean values still verifies the Legendre transform structure, leading to the standard results of thermodynamic relation [15,16]. Therefore, at least in our case of the self-gravitating system described by the distribution function f, the analysis using standard mean values can be validated and the conclusion remains correct.

The framework of nonextensive statistics based on Tsallis' type entropy seems to give a consistent generalization of the usual thermodynamical structure [14,17–19,23]. These works, however, are mainly concerned with construction of a consistent formal framework. In the light of this, our specific application of the Tsallis-type entropy to the self-gravitating systems is the first realistic physical consideration. Due to the spherical symmetry, the system is simple and we can easily evaluate physical quantities, e.g., pressure, energy and entropy. Nevertheless, the result still shows a very interesting phenomenon, i.e., the gravothermal instability. Furthermore a fact of the gravity being a long-range force suggests that the self-gravitating system is one of the most preferable and interesting testing grounds for the framework of nonextensive statistics.

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Appendix A. Some formulae for integration

Here we list some formulae which have been used in Sections 3 and 4. From the hydrostatic Eqs. (20) and (21) and the integration by parts, we obtain

$$\int_0^{r_e} \mathrm{d}r\, \frac{Gm}{r^2}\, 4\pi r^3 \rho = -\int_0^{r_e} \mathrm{d}r\, 4\pi r^3\, \frac{\mathrm{d}P}{\mathrm{d}r} = -4\pi r_e^3 P_e + 3\int_0^{r_e} \mathrm{d}r\, 4\pi r^2 P\,, \tag{A.1}$$

$$\begin{aligned}\int_0^{r_e} \mathrm{d}r\, \frac{Gm^2}{r^2} &= -\frac{GM^2}{r_e} + 2\int_0^{r_e} \mathrm{d}r\, 4\pi r^2 \frac{Gm}{r}\rho = -\frac{GM^2}{r_e} - 2\int_0^{r_e} \mathrm{d}r\, 4\pi r^3 \frac{\mathrm{d}P}{\mathrm{d}r} \\ &= -\frac{GM^2}{r_e} - 8\pi r_e^3 P_e + 6\int_0^{r_e} \mathrm{d}r\, 4\pi r^2 P\,. \end{aligned} \tag{A.2}$$

Similarly, we can rewrite the integration of pressure P as

$$\begin{aligned}\int_0^{r_e} \mathrm{d}r\, 4\pi r^2 P &= \frac{4\pi}{3} r_e^3 P_e - \int_0^{r_e} \mathrm{d}r\, \frac{4\pi}{3} r^3 \frac{\mathrm{d}P}{\mathrm{d}r} = \frac{4\pi}{3} r_e^3 P_e + \frac{G}{6}\int_0^{r_e} \frac{\mathrm{d}r}{r} \frac{\mathrm{d}}{\mathrm{d}r}(m^2) \\ &= \frac{4\pi}{3} r_e^3 P_e + \frac{GM^2}{6r_e} - \frac{n+1}{6}\int_0^{r_e} \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{P}{\rho}\right) m \\ &= \frac{4\pi}{3} r_e^3 P_e + \frac{GM^2}{6r_e} - \frac{n+1}{6}\left\{\frac{MP_e}{\rho_e} - \int_0^{r_e} \mathrm{d}r\, 4\pi r^2 P\right\}\,, \end{aligned}$$

which leads to

$$\int_0^{r_e} \mathrm{d}r\, 4\pi r^2 P = -\frac{1}{n-5}\left\{8\pi r_e^3 P_e - (n+1)\frac{MP_e}{\rho_e} + \frac{GM^2}{r_e}\right\}\,. \tag{A.3}$$

Appendix B. Second variation of entropy

In this appendix, we derive the second variation of entropy around the equilibrium state $\delta S_q = 0$.

We first express the entropy (2) of the extremum state. Substitution of the distribution function (16) leads to

$$S_q^{(\max)} = -\frac{1}{q-1}\int d^3\boldsymbol{x}\, d^3\boldsymbol{v}\,(f^q - f)$$
$$= \left(n - \frac{3}{2}\right)\left[M - \frac{\tilde{B}_n}{T^{(3/2)/(n-3/2)}}\int d^3\boldsymbol{x}\,\rho^{1+1/n}\right], \qquad \text{(B.1)}$$

where the constant $\tilde{B}_n$ is given by

$$\tilde{B}_n = \frac{B(3/2, n+1/2)}{\{B(3/2, n-1/2)\}^{(n-1/2)/(n-3/2)}}\left[4\sqrt{2}\pi\left(n - \frac{3}{2}\right)^{3/2}\right]^{-1/(n-3/2)}. \qquad \text{(B.2)}$$

Just for convenience, we introduce the quantity W

$$W = \int d^3\boldsymbol{x}\,\rho^{1+1/n} \qquad \text{(B.3)}$$

and vary (B.1) with respect to ρ and T keeping E and M fixed. Up to the second order, we obtain

$$\delta S_q = -\frac{\tilde{B}_n}{T^{(3/2)/(n-3/2)}}\left[-\frac{3}{2}\left\{\frac{W\delta T + \delta T\delta W}{T} - \left(\frac{n}{n-3/2}\right)\frac{W}{2T^2}(\delta T)^2\right\}\right.$$
$$\left. + \left(n - \frac{3}{2}\right)\delta W\right]. \qquad \text{(B.4)}$$

On the other hand, the constraint $\delta E = 0$ gives

$$0 = \delta E = \delta\left\{\frac{3}{2}\left(\frac{n-3/2}{n+1}\right)TW + \frac{1}{2}\int d^3\boldsymbol{x}\,\rho\Phi\right\}$$
$$= \frac{3}{2}\left(\frac{n-3/2}{n+1}\right)\delta(TW) + \int d^3\boldsymbol{x}\left(\Phi\delta\rho + \frac{1}{2}\delta\rho\delta\Phi\right). \qquad \text{(B.5)}$$

We thus find

$$\delta T(W + \delta W) = -\frac{2}{3}\left(\frac{n+1}{n-3/2}\right)\int d^3\boldsymbol{x}\left(\Phi\delta\rho + \frac{1}{2}\delta\rho\delta\Phi\right) - T\delta W. \qquad \text{(B.6)}$$

(In arriving at Eq. (B.5), we have used the fact that $\int d^3\boldsymbol{x}\,\Phi\delta\rho = \int d^3\boldsymbol{x}\,\rho\delta\Phi$). Substituting (B.6) into (B.4), we eliminate the quantity δT. Then, collecting the second order terms yields the second variation of entropy $\delta^2 S_q$ as follows:

$$\delta^2 S_q = -\frac{\tilde{B}_n}{T^{(3/2)/(n-3/2)}}\left[\frac{n+1}{n-3/2}\int d^3\boldsymbol{x}\,\frac{\delta\rho\delta\Phi}{2T} + n\delta W\right.$$
$$\left. + \frac{3}{4}\frac{1}{T^2 W}\frac{n}{n-3/2}\left\{\frac{2}{3}\frac{n+1}{n-3/2}\int d^3\boldsymbol{x}\,\Phi\delta\rho + T\delta W\right\}^2\right]. \qquad \text{(B.7)}$$

Now consider the variation of W:

$$\delta W = \delta\left(\int d^3\boldsymbol{x}\,\rho^{1+1/n}\right) = \frac{n+1}{n}\int d^3\boldsymbol{x}\,\rho^{1/n}\left(\delta\rho + \frac{1}{2n}\frac{(\delta\rho)^2}{\rho}\right). \tag{B.8}$$

Neglecting the higher order contributions and keeping the second-order terms only, substitution of (B.8) finally leads to expression (36):

$$\delta^2 S_q = -\frac{\tilde{B}_n}{T^{(3/2)/(n-3/2)}}\frac{n+1}{n-3/2}\left[\int d^3\boldsymbol{x}\left\{\frac{\delta\rho\delta\Phi}{2T} + \frac{n-3/2}{2n}\frac{(\delta\rho)^2}{\rho^{1-1/n}}\right\}\right.$$
$$\left. + \frac{n(n+1)}{(n-3/2)^2}\frac{1}{3T^2 W}\left\{\int d^3\boldsymbol{x}\left(\Phi + \frac{3}{2}\frac{n-3/2}{n}T\rho^{1/n}\right)\delta\rho\right\}^2\right]. \tag{B.9}$$

Note that the above equation indeed reduces to the well-known result in the isothermal sphere. Using the fact that $\tilde{B}_n \to 1$ and $W \to M$ in the limit $n \to \infty$, we obtain

$$\delta^2 S_q \stackrel{n\to\infty}{\to} -\int d^3\boldsymbol{x}\left\{\frac{\delta\rho\delta\Phi}{2T} + \frac{(\delta\rho)^2}{\rho}\right\} - \frac{1}{3T^2 M}\left\{\int d^3\boldsymbol{x}\Phi\delta\rho\right\}^2,$$

which is equivalent to expression (16) of Antonov [1] and equation (A1.10) of Padmanabhan [5].

Appendix C. Positivity of minimum eigenvalue $\kappa_{\min}$

For the sake of completeness, in this appendix, we will prove that for sufficiently small r_e, minimum eigenvalue of Eq. (45), $\kappa_{\min}$ can become positive, irrespective of the polytrope index n.

First, recall that from (42) and (45), positivity $\kappa_{\min} > 0$ is equivalent to the condition $A[Q] < 0$. This gives

$$-\int_0^{r_e} dr\left\{\frac{\delta\Phi}{2T}\frac{dQ}{dr} + \frac{n-3/2}{n}\frac{1}{8\pi r^2\rho^{1-1/n}}\left(\frac{dQ}{dr}\right)^2\right\}$$
$$-\frac{1}{3T^2 W}\frac{n(n+1)}{(n-3/2)^2}\left[\int_0^{r_e} dr\left\{\Phi + \frac{3}{2}\left(\frac{n-3/2}{n}\right)T\rho^{1/n}\right\}\frac{dQ}{dr}\right]^2 < 0. \tag{C.1}$$

In the above expression, while the second term in the right-hand side of equation is always negative, the first term can be expressed as

$$[\text{1st term}] = -\frac{1}{2T}(H-1)\int_0^{r_e} dr\,\frac{GQ^2}{r^2}, \tag{C.2}$$

where we define

$$H \equiv \frac{\int_0^{r_e} dr\,1/4\pi r^2\rho\,(P/\rho)(dQ/dr)^2}{n/(n+1)\int_0^{r_e} dr\,GQ^2/r^2}. \tag{C.3}$$

Thus, the inequality $H > 1$ provides a sufficient condition for the positivity of $\kappa_{\min}$.

Now, we rewrite the definition (C.3) integrating by parts. We obtain

$$-\frac{d}{dr}\left\{\frac{1}{4\pi r^2\rho}\left(\frac{P}{\rho}\right)\frac{dQ}{dr}\right\} = H\frac{n}{n+1}\frac{GQ}{r^2}\,. \tag{C.4}$$

The above equation can be regarded as the eigenvalue equation with eigenvalue H. In fact, one can show that Eq. (C.4) has minimum eigenvalue $H_{\rm min} = 1$ for some small radius r_e. Consider the function

$$Q_{\rm min} = c\left\{4\pi r^3\rho(r) - \frac{n-3}{n-1}m(r)\right\} = c\left(u - \frac{n-3}{n-1}\right)m\,, \tag{C.5}$$

where c is merely a constant. The above function behaves like $Q_{\rm min} \simeq c(2n/(n-1))m \to 0$ near the origin $r = 0$ and it also becomes vanishing at $r = r_1$, where $u(r_1) = (n-3)/(n-1)$. Further, $Q_{\rm min}$ does not vanish during the interval $(0, r_1)$. Hence, when setting the radius of the wall as $r_e = r_1$, the function (C.5) indeed corresponds to the ground state of the eigenvalue equation, i.e., perturbation mode without any nodes. Therefore, this fact proves that if we suitably choose a smaller radius $r_e < r_1$, the eigenvalue of system (C.4) should be larger than unity, i.e., $H > 1$. Hence, all equilibrium configurations with $r_e < r_1$ become stable.

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