The Applications of Generalized Entropies and Generalized Variational Principle to Long-range Interacting Systems

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Self-gravitating Stellar System

A System with N Particles (stars) (N>>1)

Particles interact with Newtonian gravity each other

\[ H = \sum_i \frac{\vec{p}_i^2}{2m_i} - \sum_i \sum_{j \neq i} \frac{Gm_im_j}{|\vec{x}_i - \vec{x}_j|} \]

Typical example of a system under long-range force

Key word **Negative Specific Heat**

Long-term (thermodynamic) instability \( (t > t_{\text{relax}}) \)

Gravothermal instability

\[ \text{Antonov 1962} \]
\[ \text{Lynden-Bell & Wood 1968} \]
**Time scales of Self-gravitating System**

**free-fall time**

\[ T_{\text{free}} \approx \frac{1}{\sqrt{G\rho}} \]

Dynamical time scale

**two-body relaxation time**

\[ T_{\text{relax}} \approx 0.1 \frac{N}{\ln N} T_{\text{free}} \]

time scale for approaching to thermal equilibrium

Motion driven by mean gravitational potential

Mean mass density

Time scale for loss of memory by two-body collision

relaxation
For systems with negative specific heat,

Which is unstable under the condition $E, M=\text{const.}$?
The system with larger $r_e$ becomes unstable

Critical Radius

$$r_c = 0.335 \frac{GM^2}{(-E)}$$

$M$ Total mass

$E$ Total energy

Unstable

Stable

Large $r_e > r_c$

Small $r_e < r_c$

$D = \rho_c / \rho_e > 709$

$D = \rho_c / \rho_e < 709$

$\rho_c$ density at center $r = 0$

$\rho_e$ density at wall $r = r_e$

Extended Halo

Concentrated Core
Maxwell-Boltzmann distribution as Equilibrium

**Boltzmann-Gibbs entropy**

\[ S_{BG} = -\int d^3x \, d^3v \, f(x, v) \ln f(x, v) \]

**Maxwell-Boltzmann distribution**

\[ f(x, v) \propto \exp(-\beta \varepsilon) ; \quad \varepsilon = v^2 / 2 + \Phi(x) \]

Extremization

\[ 0 = \delta S_{BG} - \alpha \delta M - \beta \delta E \]

No stable isothermal state exists at

\[ \lambda = -\frac{r_e E}{GM^2} \]

- **Large** \( r_e \) : \( r_e > \lambda_{crit} \frac{GM^2}{-E} \)
- **High density-contrast** : \( D > D_{crit} \)

\[ D_{crit} = 709 \]

\[ \lambda_{crit} = 0.335 \]
A naïve generalization of BG statistics

Thermostatistical treatment by generalized entropy

\[ S_q = -\frac{1}{q-1} \int d^3x d^3v \left[ \{p(x, v)\}^q - p(x, v) \right] \]

\( q \)-entropy

One-particle distribution function identified with \textit{escort distribution}

\[ f(x, v) = M \frac{\{p(x, v)\}^q}{\int d^3x d^3v \{p(x, v)\}^q} \]

Power-law distribution

\[ f(x, v) = A \left[ \mathcal{E}_0 - \mathcal{E} \right]^{q/(1-q)} \]

\( \mathcal{E} = \frac{v^2}{2} + \Phi(x) \)

Stellar polytrope as quasi-equilibrium state

This power-law type distribution is well-known in subject of stellar dynamics, referred to as

“stellar polytrope”

(e.g., Binney & Tremaine 1987)

Polytropic equation of state

\[ P(r) = K_n \rho^{1+1/n}(r) \]

Polytrope index

\[ n = \frac{1}{1-q} + \frac{1}{2} \begin{cases} n \to \infty \\ \text{BG limit} \end{cases} \]
Stellar polytrope as quasi-equilibrium state

Energy-density contrast relation for stellar polytrope

oscillatory behavior appears when $n>5$.

For larger density $D>D_{\text{crit}}$, unstable state appears at $n>5$.

(gravothermal instability)
Survey results of group (A)

The evolutionary track keeps the direction increasing the polytrope index “$n$”.

Once exceeding the critical value “$D_{\text{crit}}$“, central density rapidly increases toward the core collapse.
Run n3A: N-body simulation

Initial cond: Stellar polytrope (n=3, D=10)

Density profile

One-particle distribution function

Fitting to stellar polytropes is quite good until $t \sim 30 \, t_{rh,i}$. 

$\epsilon = \frac{1}{2} v^2 + \Phi(x)$
Overview of the N-body results

Stellar polytropes are not stable in timescale of two-body relaxation. However, focusing on their transients, we found:

Quasi-equilibrium property

- Transient states approximately follow a sequence of stellar polytropes with gradually changing polytrope index “n”.

Quasi-attractive behavior

- Even starting from non-polytropic states, system soon settles into a sequence of stellar polytropes.
Polytrope describes transient states of the self-gravitating system.

**Question:** Other examples in systems with long-range interactions?

**Answer:** Yes

2D HMF model
Hamiltonian Mean-Field (HMF) Model

**Hamiltonian**

\[ H_N(\theta, p) = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + \frac{1}{2N} \sum_{j,k=1}^{N} [1 - \mathbf{m}_j \cdot \mathbf{m}_k] \]

\[ \mathbf{m}_j = (\cos \theta_j, \sin \theta_j) \]

**θ-space**

**Potential**

**Mean-Field & Canonical Equations of Motion**

\[ M = \frac{1}{N} \sum_{j=1}^{N} \mathbf{m}_j = (M_x, M_y) = (M \cos \phi, M \sin \phi) \]

\[ \frac{d\theta_j}{dt} = p_j, \quad \frac{dp_j}{dt} = -M \sin(\theta_j - \phi) \]
2D HMFモデル

\[ H = \frac{1}{2} \sum_{i=1}^{N} (p_{x,i}^2 + p_{y,i}^2) + \frac{1}{2N} \sum_{i,j}^{N} \left[ 3 - \cos(x_i - x_j) - \cos(y_i - y_j) - \cos(x_i - x_j)\cos(y_i - y_j) \right] \]

Interaction by Mean-field: **Long-range** interacting system

2D HMF have the effect of **energy transfer** due to **2-body scattering process**.

**Negative specific heat** in some range of energy
$U = 1.95$, $N = 10^4$, $N = 9 \times 10^4$

**T-U curve**  **Boltzmann case**

**U (energy)**

**Magnetization**

**Vlasov phase?**

**dist. func.**
Initial: polytrope $U=1.9$

Negative specific heat

$M$ (logarithmic)

$|P|$

Projected density

$E$ $D$ $C$ $B$ $A$

$E$ $D$ $C$ $B$ $A$
Initial: polytrope  $U=1.7$

positive specific heat

$M_{\text{dist. func.}}$

$t$ (logarithmic)
Initial WB $U=1.9$

Negative specific heat

![Graph of Magnetization vs. t (logarithmic)]

![Graph of Projected density vs. x]

![Graph of Distribution function]
Initial WB $U=1.7$

positive specific heat

Magnetization

$t$ (logarithmic)

dist. func.

$|p|$
How to derive the evolution equation for polytropic index, q or n.

self-gravitating systems
Kinetic-theory approach

For a better understanding of the quasi-equilibrium states,

**Fokker-Planck (F-P) model for stellar dynamics**

**orbit-averaged F-P eq.**

\[
\left( \frac{\partial f(\varepsilon)}{\partial t} \right)_\tau = \Gamma \left( \frac{\partial \Pi(\varepsilon)}{\partial \tau} \right)_t ; \quad \Gamma = 16\pi^2 G^2 m^2 \ln \Lambda
\]

\[
\Pi(\varepsilon) = \int d\varepsilon' f(\varepsilon)f(\varepsilon') \min[\tau(\varepsilon),\tau(\varepsilon')] \left\{ \frac{\partial \ln f(\varepsilon)}{\partial \varepsilon} - \frac{\partial \ln f(\varepsilon')}{\partial \varepsilon'} \right\}
\]

\[
\tau(\varepsilon) = \frac{16\pi^2}{3} \int dr r^2 \left\{ 2[\varepsilon - \phi(r)] \right\}^{3/2}
\]

Complicated, but helpful for semi-analytic understanding
Generalized Variational Principle for F-P eq.

Local potential

$$\Phi(f, f_0) = \int d\tau \left( \frac{\partial f_0}{\partial t} \right) \ln f + \frac{\Gamma}{4} \int \int d\epsilon d\epsilon' \ f_0 f_0' \min(\tau_0, \tau_0') \left( \frac{\partial \ln f}{\partial \epsilon} - \frac{\partial \ln f'}{\partial \epsilon'} \right)^2$$

- Variation w.r.t. $f$ $\frac{\delta}{\delta f} \Phi(f, f_0) \bigg|_{f=f_0} = 0$ F-P equation for $f_0$
- $f_0$ fixed
- Absolute minimum at a solution $f_0$

$$\Delta \Phi \equiv \Phi(f, f_0) - \Phi(f_0, f_0) > 0$$

Application: Takahashi & Inagaki (1992); Takahashi (1993ab)

Glansdorff & Prigogine (1971)
Inagaki & Lynden-Bell (1990)
The evolution eq. for “$n$” from generalized variational principle

Assuming stellar polytropes with time-varying polytrope index as transient state,

**trial function**

$$f(\varepsilon) = A(t)[\varepsilon_0(t) - \varepsilon]^{n(t)-3/2} \quad (A \text{ and } \varepsilon_0 \text{ are the functions of } n)$$

$$\left. \frac{\delta}{\delta n} \Phi(f,f_0) \right|_{f=f_0} = 0$$

**function of** $n, E, M$

- \[ \dot{n}(t) = - \left( \frac{\partial \lambda}{\partial \xi_e} \right)_n \int d\varepsilon \Pi(\varepsilon) \left\{ \left( \frac{\partial \lambda}{\partial \xi_e} \right)_n \left( \frac{\partial}{\partial n} \right)_{\xi_e} - \left( \frac{\partial \lambda}{\partial n} \right)_{\xi_e} \left( \frac{\partial}{\partial \xi_e} \right)_n \right\} \frac{\partial \ln f}{\partial \varepsilon} \]

- \[ \int d\tau f(\varepsilon) \left\{ \left( \frac{\partial \lambda}{\partial \xi_e} \right)_n \left( \frac{\partial \ln f}{\partial n} \right)_{\xi_e} - \left( \frac{\partial \lambda}{\partial n} \right)_{\xi_e} \left( \frac{\partial \ln f}{\partial \xi_e} \right)_n \right\}^2 \]
Semi-analytic prediction: evolution of “n”

Time-scale of quasi-equilibrium states is successfully reproduced from semi-analytic approach based on variational method.
Summary and Discussion

(1) Polytrope (Extremum of Generalized Entropy)
   \[ \text{Transient states to thermal equilibrium} \]
   Self-gravitating system, 2D-HMF \text{ Negative specific heat}
   Long-range interaction

(2) Generalized variational principle for F-P eq.
   \[ \text{Evolution eq. for polytropic index} \]

Works in Progress: Short-range attracting interaction
   \text{negative specific heat}
   Polytrope ? \text{ Superposition of Boltzmann dist. ?}
Summary

(1) 重力多体系
長距離力（引力）
比熱が負
small system

(2) 準定常状態
ポリトロープ状態の系列で記述できる

\[ P(r) = K_n \rho^{1+1/n}(r) \]
\[ n = \frac{1}{1-q} + \frac{1}{2} \]

(3) ポリトロープ指数 n の時間発展
一般化された変分原理
Fokker-Planck eq.
ポリトロープ状態: Trial func

非平衡進化
準定常状態が存在
(1) ポリトロープ ⇔ 準定常状態：他の例はあるか

2次元HMFモデルの解析

(2) ポリトロープ ⇔ 準定常状態：長距離相互作用が本質？

Yukawa型相互作用での解析

(3) ポリトロープによる準定常状態の記述の限界

ポリトロープは core collapse 前しか適用できない？